Problem 1.7

When the temperature of liquid mercury increases by one degree Celsius (or one kelvin), its volume increases by one part in 5500. The fractional increase in volume per unit change in temperature (when the pressure is held fixed) is called the **thermal expansion coefficient**, β:

$$\beta \equiv \frac{\Delta V/V}{\Delta T}$$

(where V is volume, T is temperature, and Δ signifies a change, which in this case should really be infinitesimal if β is to be well defined). So for mercury, $\beta = 1/5500\ \text{K}^{-1} = 1.81 \times 10^{-4}\ \text{K}^{-1}$. (The exact value varies with temperature, but between 0°C and 200°C the variation is less than 1%.)

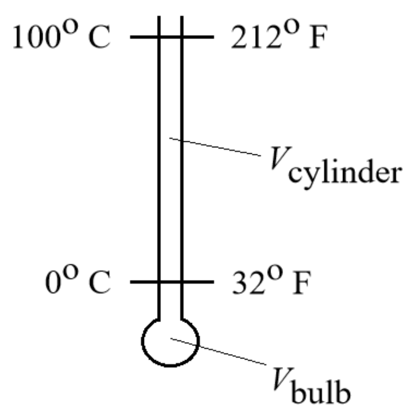
(a) Get a mercury thermometer, estimate the size of the bulb at the bottom, and then estimate what the inside diameter of the tube has to be in order for the thermometer to work as required. Assume that the thermal expansion of the glass is negligible.

(b) The thermal expansion coefficient of water varies significantly with temperature: It is $7.5 \times 10^{-4}\ \text{K}^{-1}$ at 100°C, but decreases as the temperature is lowered until it becomes *zero* at 4°C. Below 4°C it is slightly *negative*, reaching a value of $-0.68 \times 10^{-4}\ \text{K}^{-1}$ at 0°C. (This behavior is related to the fact that ice is less dense than water.) With this behavior in mind, imagine the process of a lake freezing over, and discuss in some detail how this process would be different if the thermal expansion coefficient of water were always positive.

Solution

Part (a)

A schematic of a thermometer is drawn below. It consists of a bulb and a cylinder.



The bulb is very small, with dimensions of millimeters. A reasonable estimate of the volume is then 0.1 cm^3. The length of the cylinder (from the 0°C mark to the 100°C mark) is about 20 cm.

With these numbers and the given formula for the thermal expansion coefficient, the radius of the cylinder can be calculated.

$$\begin{aligned}\beta &= \frac{\Delta V/V}{\Delta T} \\ &= \frac{\Delta V}{V\Delta T} \\ &= \frac{V_f - V}{V(T_f - T)} \\ &= \frac{(V_{\text{bulb}} + V_{\text{cylinder}}) - V_{\text{bulb}}}{V_{\text{bulb}}(T_f - T)} \\ &= \frac{V_{\text{cylinder}}}{V_{\text{bulb}}(T_f - T)}\end{aligned}$$

Solve for V_{cylinder}.

$$V_{\text{cylinder}} = \beta V_{\text{bulb}}(T_f - T)$$

Substitute the formula for the volume of a cylinder.

$$\pi r^2 L = \beta V_{\text{bulb}}(T_f - T)$$

Solve for r.

$$r = \sqrt{\frac{\beta V_{\text{bulb}}(T_f - T)}{\pi L}}$$

The diameter is twice the radius. Note that $T_f - T$ is the temperature difference in kelvin over the length of the cylinder.

$$\begin{aligned}d = 2r &= 2\sqrt{\frac{\beta V_{\text{bulb}}(T_f - T)}{\pi L}} \\ &\approx 2\sqrt{\frac{\left(\frac{1}{5500\text{ K}}\right)(0.1\text{ cm}^3)(373.15\text{ K} - 273.15\text{ K})}{\pi(20\text{ cm})}} \\ &\approx 0.01\text{ cm}\end{aligned}$$

Therefore, the diameter of the cylinder containing the mercury is about 0.1 mm.

Part (b)

A negative thermal expansion coefficient between 0°C and 4°C means that the volume of water increases as the temperature decreases. In other words, as the temperature lowers from 4°C to 0°C, the density lowers as well, making the ice-cold water rise to the surface. Warmer water sinks below the surface, which is critical for the existence of marine life. Ice has a smaller density than water, making it float on the surface. This helps to insulate the water from the far colder weather above. As the thickness of the ice increases, the insulation from the weather increases as well. If a body of water is shallow, the warmer water will have nowhere to sink to; the body of water will freeze entirely. If the thermal expansion coefficient of water were always positive, colder water would take up less space (as in a thermometer), making it sink to the bottom because of its larger density. Warmer water would rise to the surface because of its lower density and be cooled from the mixing that occurs in addition to the cold weather. Lakes would freeze more easily as a result.